Posets and their Incidence Algebras

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Outline

What is a poset?

Definition

A **poset** $(P, \preceq)$ is a set P which has a **partial order** $\preceq$ imposed on it, and $\preceq$ is reflexive, antisymmetric, and transitive. That is,

- For $t \in P, t \preceq t$.
- $s \preceq t$ and $t \preceq s$ means $t = s$.
- $s \preceq t$ and $t \preceq u$ means $s \preceq u$.

Note. This does not mean that for all $s, t \in P$ we have $s \preceq t$ or $t \preceq s$. There need not be a comparison under $\preceq$, and when this is the case we call s, t incomparable. This is shown by writing $s||t$.

Drawing a poset

Definition

The **Hasse diagram** of a poset is a graph where $s \in P$ is a vertex, and s, t are connected by an edge if $s \preceq t$ and there is no u such that $s \prec u \prec t$. If $s \preceq t$, s is placed above t in the Hasse diagram.

We can say that posets P and Q are **isomorphic** if we can create an order preserving bijection ϕ between them. This is denoted $P \cong Q$.

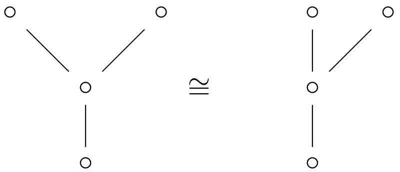


Figure 1: Isomorphic Hasse diagrams of a 4-element poset.

Basic terms in a poset

Definition

A closed **interval** $[a, b]$ is the set of all $p \in P$ such that $a \preceq p \preceq b$. The set of closed intervals is denoted $\mathrm{Int}(P)$.

Definition

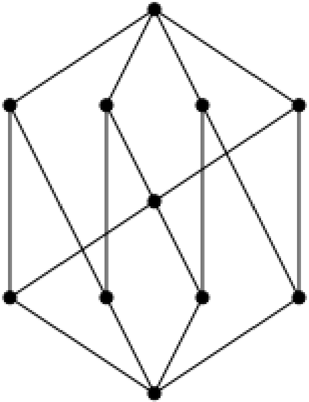
A **chain** is a collection of elements t_i of a poset P such that $t_0 \prec t_1 \ldots \prec t_n$.

Definition

A **multichain** s a collection of elements t_i of a poset P such that $t_0 \preceq t_1 \ldots \preceq t_n$.

Examples

What do chains and intervals look like in a Hasse diagram?



The Incidence Algebra

Definition

The **incidence algebra** on a poset P over a field K, denoted $I(P, K)$ is a K-algebra over the vector space of functions

$$f : \text{Int}(P) \to K$$

equipped with the bilinear product called convolution, denoted $*$, given by

$$f * g([s, u]) := \sum_{s \preceq t \preceq u} f([s, t])g([t, u]).$$

Note. It is usually not necessary to use fields where $\text{ch}(K) \neq 0$, so using $K = \mathbb{C}$ is often sufficient when working with $I(P, K)$. From this point on, we will use $I(P, \mathbb{C})$.

The identity in $I(P, \mathbb{C})$

The delta function δ is given by

$$\delta([s,t]) = \begin{cases} 1, s = t \\ 0, s \neq t \end{cases} .$$

It is a two sided identity under convolution: $f * \delta = \delta * f = f$.

The zeta and Möbius functions

The **zeta** function ζ is given by

$$\zeta([s,t]) = 1.$$

We define the **Möbius** function μ is defined by

$$\mu * \zeta = \delta.$$

Proposition

The function μ can be computed by setting $\mu([s,s]) = 1$, and

$$\mu([s,u]) = -\sum_{s \preceq t \prec u} \mu([s,t]).$$

Why does the zeta function matter?

One major reason is that the zeta function is extremely useful for counting chains. Denote f^n as $f * f \ldots * f$ n times for $f \in I(P, \mathbb{C})$. Then we have the following propositions:

Proposition

$$\zeta^n([s,t]) = \sum_{s \preceq s_1 \cdots \preceq s_{n-1} \preceq t} 1.$$

Proposition

The function $(\zeta - \delta)^n([s,t])$ counts the number of chains in the interval $[s,t]$.

Möbius Inversion Theorem

Theorem (Möbius Inversion)

Let P be a poset where for every $t \in P$ the order ideal Λ_t is finite. For $f, g : P \to \mathbb{C}$ we have

$$g(t) = \sum_{s \preceq t} f(s)\zeta([s,t]) \Leftrightarrow f(t) = \sum_{s \preceq t} g(s)\mu([s,t])$$

for all $t \in P$.

Note. The order ideal Λ_t is defined as the set of elements less than t.

Proving the theorem

Take $\mathbb{C}^P$, the set of functions $f : P \to \mathbb{C}$. Then $I(P, \mathbb{C})$ acts on $\mathbb{C}^P$ via

$$(f\mathcal{I})(t) = \sum_{s \preceq t} f(s)\mathcal{I}([s, t])$$

for $f \in \mathbb{C}^P, \mathcal{I} \in I(P, \mathbb{C})$. Then we have the equivalent statement to Möbius inversion

$$g = f\zeta \Leftrightarrow f = g\mu.$$

Note. Here, $I(P, \mathbb{C})$ acts on the right. We get a 'dual' theorem by acting on the left.

Example

Let B_n be the poset with underlying set $\{S \mid S \subseteq [n]\}$ and partial order $\preceq$ given by $S \preceq T$ if $S \subseteq T$. We then have

$$\mu_{B_n} = (-1)^{|T-S|}.$$

Applying Möbius inversion, we obtain

$$g(T) = \sum_{S \subseteq T} f(S) \Leftrightarrow f(T) = \sum_{S \subseteq T} g(S)(-1)^{|T-S|}.$$

Setting $f(T) = f_{=}(T)$ where $f_{=}$ counts objects having exactly the properties in T, and $g(T) = g_{\leq}(T)$ where $g_{\leq}$ counts objects having at most the properties in T we obtain the principle of Inclusion-Exclusion.

Möbius Inversion (number theory)

Example

Take the 'divisor poset' D_n, which has an underlying set of $\{d : d \in \mathbb{N}, d|n\}$ and partial order $\preceq$ given by $a \preceq b$ if $a|b$. We obtain that

$$\mu_{D_n}([s,t]) = \begin{cases} (-1)^k \text{ if } t/s = \prod_i p_i \text{ for distinct primes } p_i \\ 0 \text{ otherwise} \end{cases}$$

from which it follows

$$g(n) = \sum_{d|n} f(d) \Leftrightarrow f(n) = \sum_{d|n} g(d)\mu_{D_n}([d,n]).$$

Thanks

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